Monte Carlo Analysis

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Introduction

- Engineers are often asked to address the effects of uncertainty on their models
- A typical question asks: If we have uncertainty in our inputs, what is the effect on the output?
- In other words, how are our models affected if our input assumptions are incorrect?

Monte Carlo Analysis

- Monte Carlo approaches are quite useful for problems such as this
- The general idea is to sample the inputs, run a model, and thus get sampled output
- We can then look at averages, variances, probability distributions, etc.
- Business decisions can then be made from these results

More Monte Carlo

- Monte Carlo approaches are also valuable simulation approaches in themselves:
  - Particle transport
  - Random walk
  - Numerical integration (especially many-dimensional)

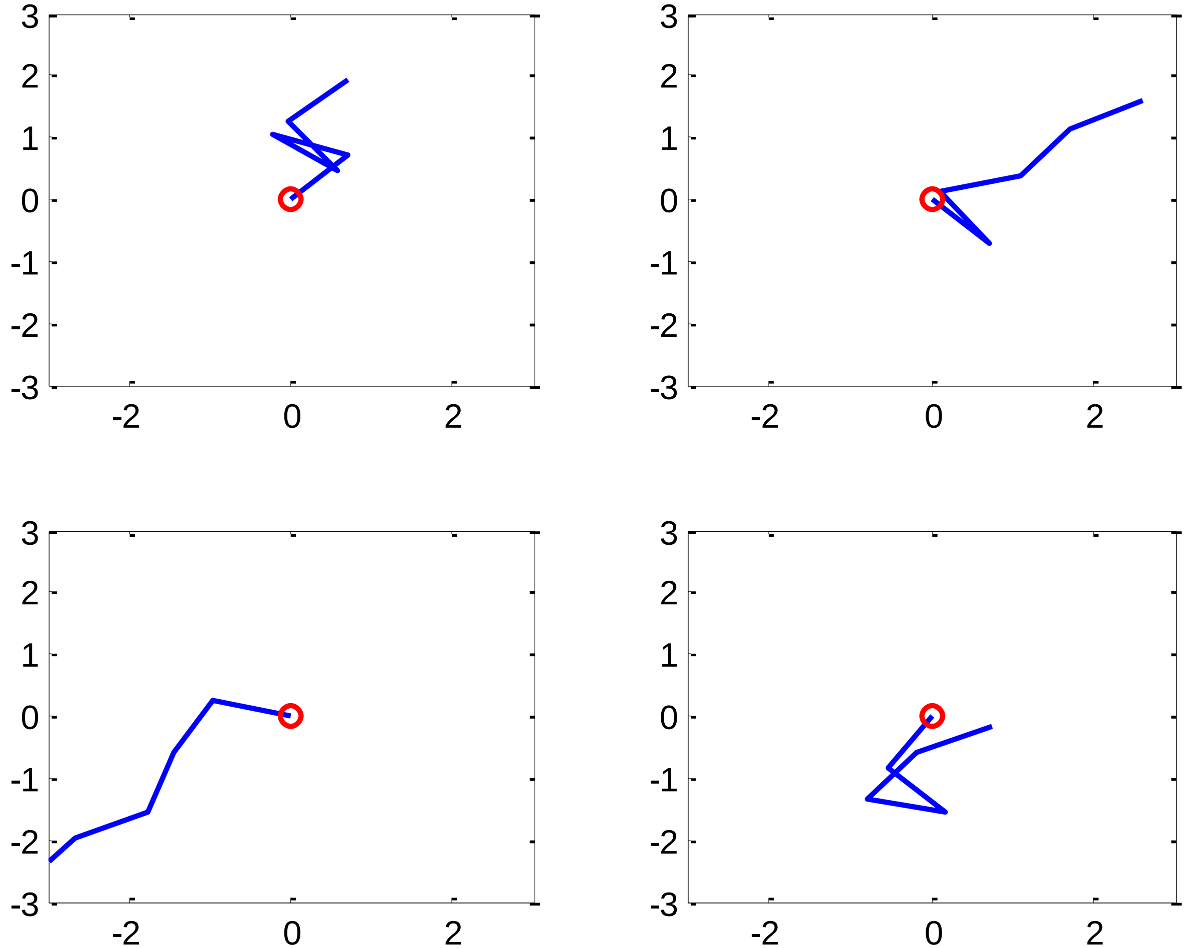
An Example

- Random walk
  - Assume path length per step is fixed
  - Randomly sample angle at which step is taken
  - Repeat many times and study resulting path
  - This is not the only algorithm for random walk. Many limit to finite number of directions and vary length from jump to jump

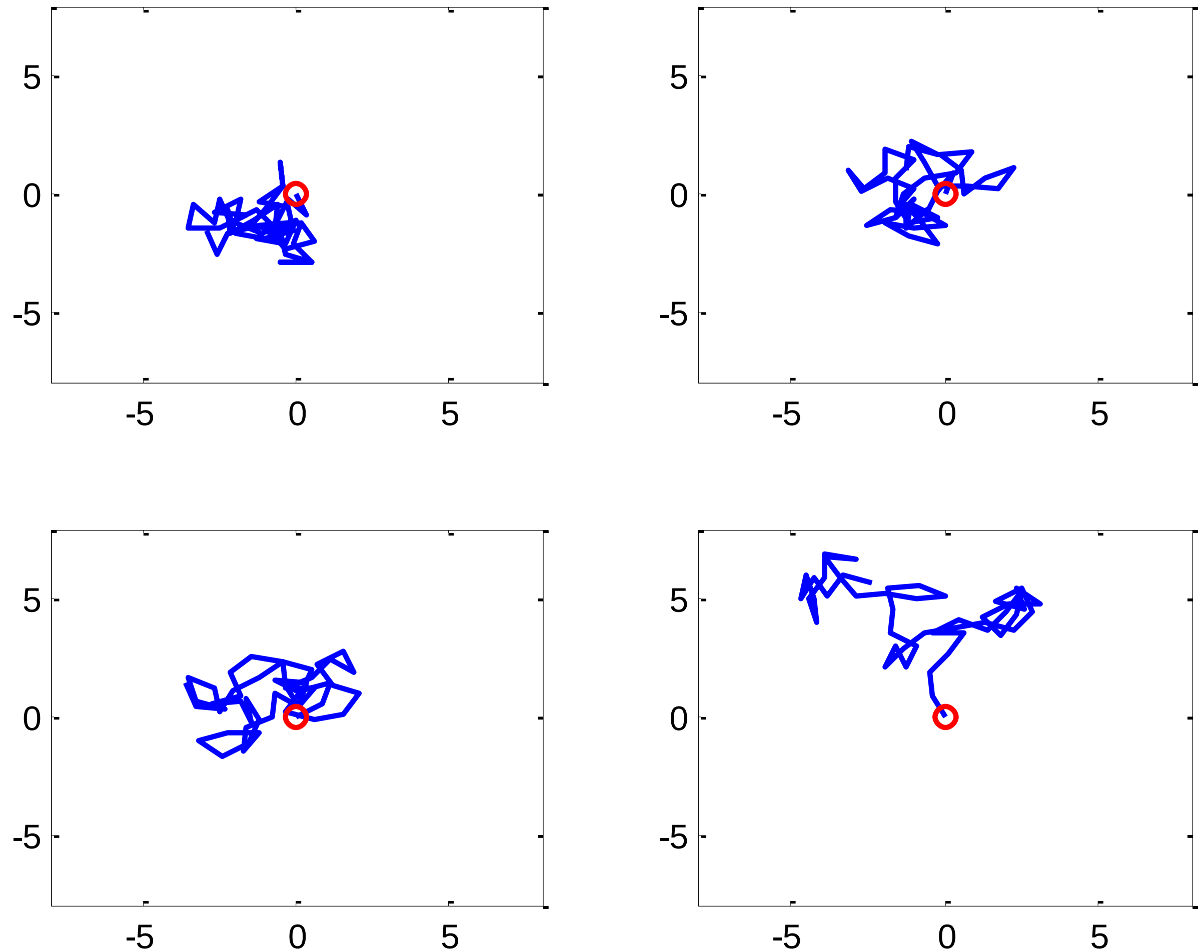
Sample

```
clear all
steplen=1;
startx=0;
starty=0;
nsteps=100;
angle=2*pi*rand(nsteps,1);
dx=steplen*cos(angle);
dy=steplen*sin(angle);
x(1)=startx;
y(1)=starty;
for i=2:nsteps
   x(i)=x(i-1)+dx(i-1);
   y(i)=y(i-1)+dy(i-1);
end
plot(x,y,0,0,'ro','LineWidth',2)
```

4 Runs for 5 Steps Each



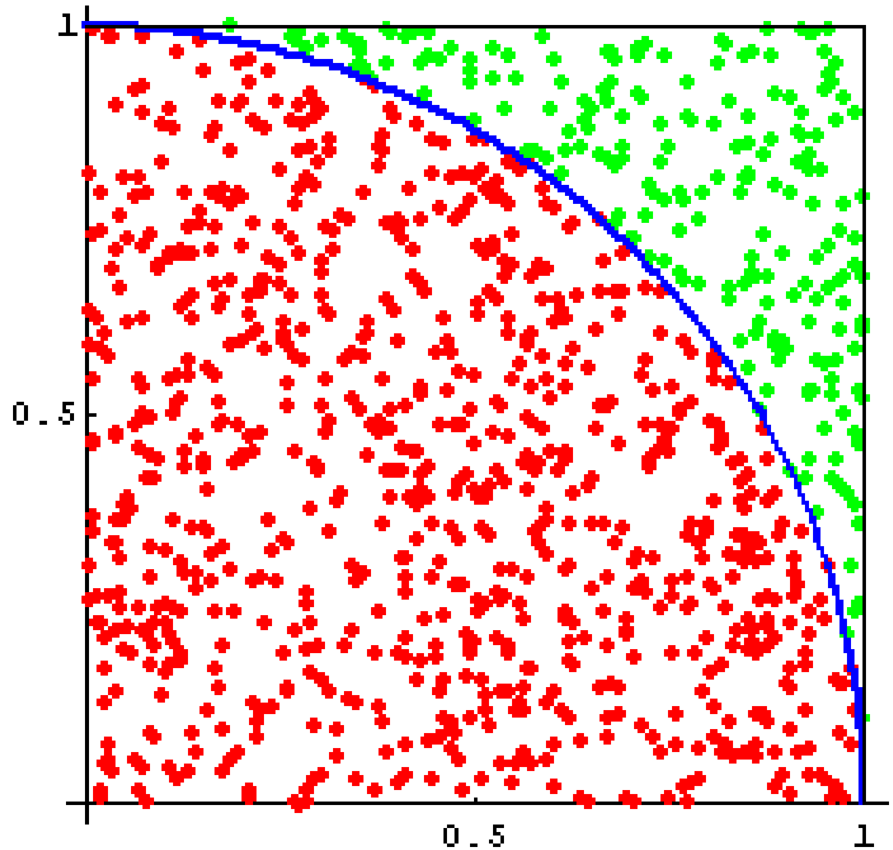
4 Runs for 50 Steps Each



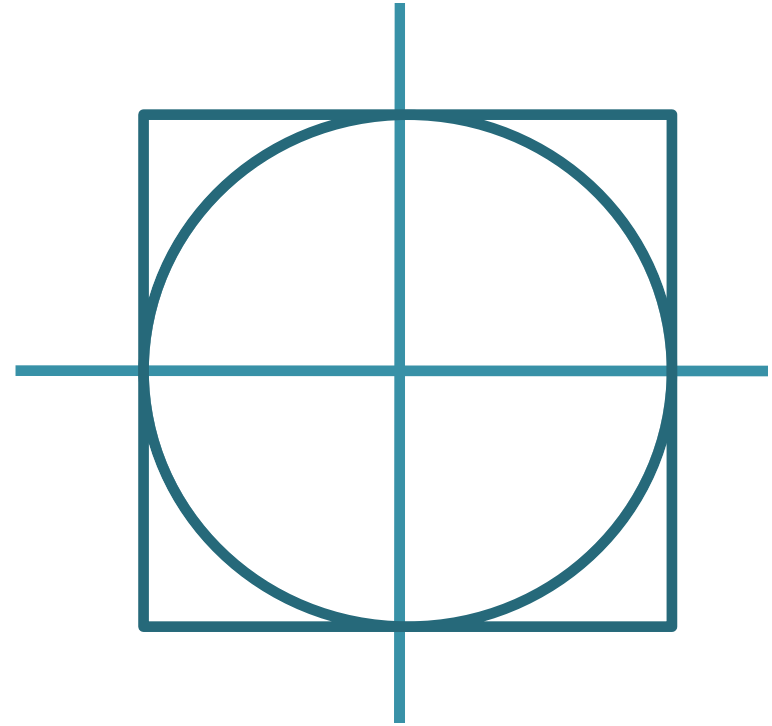
Another Example

- Numerical integration (2-D, in this case)
  - Draw area within a square
  - Randomly locate points within the square
  - Count up the number of points (N) within the area
  - Area=area of square*number points inside area/N

Finding the Area of a Circle



Example



```
clear all
squaresidelength=2;
area=squaresidelength.^2;
samples=100000;
x=squaresidelength*(-0.5+rand(samples,1));
y=squaresidelength*(-0.5+rand(samples,1));
outside=floor(2*sqrt(x.^2+y.^2)/squaresidelength);
circarea=(1-sum(outside)/samples)*area
```

Practice

- Download montecarloscripts.m
- Extract integration code
- What is area for 100 samples?
- How about 1,000 samples?
- How about 10,000?

Characteristics of Monte Carlo Approaches

- We have to sample enough times to get reasonable results
- Accuracy only increases like sqrt(N)
- Computation times are typically long
- Development time is typically relatively short
- These are a trade-off

Our Case Study

- Consider a cantilever beam of length L with a circular cross section of radius R
- The deflection of such a beam, loaded at the end, is given by

$$\Delta = \frac{FL^3}{3EI}$$

$$I = \frac{\pi R^4}{4}$$

Parameters

$$\Delta = \frac{FL^3}{3EI}$$

$$I = \frac{\pi R^4}{4}$$

- F varies from 600 N to 800 N (uniformly)
- R varies from 2 cm to 2.4 cm (uniformly)
- E varies from 185 to 200 GPa (uniformly)
- L varies from 1 m to 1.05 m (uniformly)
- What is average displacement?
- What does probability distribution look like?

Uniform Distributions

- Most codes produce random numbers (R_n) between 0 and 1 with uniform distributions
- To get a uniform distribution from a to b, you can use

$$\boxed{U = a + R_n(b-a)}$$

Normal Distributions

- These are like the well-known bell curve
- Codes often give normal distribution with mean of 0 and standard dev. of 1
- We can use the following formula to generate a normal distribution with mean of M and standard dev. of σ

$$\boxed{N = \sigma R_n + M}$$

Matlab

- Matlab has several tools for random number generation
- RAND() produces matrices of uniform numbers
- RANDN() produces matrices of random numbers with normal distributions

Using Matlab

- Put random numbers in a vector
- Use mean function

```
a=2
b=7
randnumbers=a+(b-a)*rand(5,1)
mean(randnumbers)
```

Basic Analytical Functions

- mean
- std – standard deviation
- hist(v,n) – gives histogram of set of numbers in vector v, using n bins

Practice

- Generate 1,000 random numbers uniformly distributed between 10 and 12 and calculate mean
- Repeat for 10^4, 10^5, and 10^6 samples
- Plot histogram for last case
- Note: previous code was

```
a=2
b=7
randnumbers=a+(b-a)*rand(5,1)
mean(randnumbers)
```

Practice

- What is expected value (mean) of $2*x$?
- What is the expected value of x^2?
- What is the expected value of $1/x$?

Practice

- What is the mean of 10,000,000 numbers normally distributed with a mean of 0 and standard deviation of 0?

Practice

- Complete case study for beam deflections
- Download the file beam.m and adapt to find mean deflection and histogram of deflections
- **n=100;**
- **f=600+200*rand(n,1);**
- **r=0.02+0.004*rand(n,1);**
- **emod=(185+15*rand(n,1))*1e9;**
- **l=1+0.05*rand(n,1);**
- **inert=pi*r.^4/4;**

$$\Delta = \frac{FL^3}{3EI}$$

$$I = \frac{\pi R^4}{4}$$

Another Example

- If we have 20 people in a room, what is the probability that at least two will have birthdays on the same day

Source

```
nump=23;
samples=10000;
birthd=ceil(365*rand(nump,samples));
count=0;
for j=1:samples
    if numel(birthd(:,j))-numel(unique(birthd(:,j))) >0
        count=count+1;
    end
end
probab=count/samples;
```

Practice

- If you deal 2 hands of blackjack from a fresh deck, what are the odds of the first drawing blackjack?
- Download the file vegas.m and run to find out.

vegas.m

```
function vegas
nsamples=10000;
count=0;
for i=1:nsamples
    c=randperm(52);
    points=value(c);
    aces=find(points==1);
    points(aces)=points(aces)+10;
    hand=points(1)+points(3);
    if hand==21
        count=count+1;
    end
end
twentyones=count/nsamples
```

```
function v = value(x)
v = mod(x-1,13)+1;
v = min(v,10);
```

A Modified Example

- Suppose you are dealt a hand that totals 15 and the dealer shows a face card.
- If you stay, what are your odds of winning the hand.
- Again, start from a fresh deck.
- How would you alter vegas.m to answer this question.

Questions?